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The localized induction hierarchy and the Lund–Regge equation

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Abstract. An evolution equation of a curve is constructed by summing up the infinite sequence of commuting vector fields of the integrable hierarchy for the localized induction equation (LIE). It is shown to be equivalent to the Lund–Regge equation. The intrinsic equations governing the curvature and torsion are deduced in the form of integrodifferential evolution equations. A class of exact solutions which correspond to the permanent forms of a curve evolving by a steady rigid motion are presented. The analysis of the solutions reveals that, given the shape, there are two speeds of motion, one of which has no counterpart in the case of the LIE.

1. Introduction

The motion of vortex tubes in an inviscid incompressible fluid is described by the Biot–Savart law. The localized induction equation (LIE) is the simplest model to capture the leading-order behaviour of the three-dimensional self-induced motion of a vortex filament [1, 2]. Hasimoto [3] showed that the LIE is equivalent to the cubic nonlinear Schrödinger equation (NLS) for a complex variable, implying that the LIE is completely integrable. By complete integrability, we mean that the evolution equation has an infinite sequence of independent integrals in involution. Magri [4] unveiled the bi-Hamiltonian structure that underlies this integrability and thereby manipulated a recursion operator to generate an infinite sequence of integrals in involution and commuting Hamiltonian vector fields. Langer and Perline [5] made an effort to lift the structure of the NLS to the LIE by taking advantage of the Hasimoto map. They restricted the space of curves to a class called the balanced asymptotically linear curves (BAL), and proved that the Hasimoto map is a Poisson map with respect to the appropriate Poisson structure on the BAL. Relying on this connection, they constructed a recursion operator to generate an infinite sequence of commuting vector fields associated with the LIE. We call this sequence the Langer–Perline hierarchy (LPH).

Let $\boldsymbol{X} = \boldsymbol{X}(s, t)$ be a point on the filament and $\boldsymbol{V}^{(n)} = \boldsymbol{V}^{(n)}(s, t)$ the nth term of the LPH with s and t being the arclength and the time, respectively. They are listed as follows:

$$\boldsymbol{V}^{(1)} = \boldsymbol{X}_s \times \boldsymbol{X}_{ss} \tag{1.1}$$

$$\boldsymbol{V}^{(2)} = \boldsymbol{X}_{sss} + \tfrac{3}{2}\boldsymbol{X}_{ss} \times (\boldsymbol{X}_s \times \boldsymbol{X}_{ss}) \tag{1.2}$$

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$$\vdots$$
$$\boldsymbol{V}^{(n)} = -\boldsymbol{X}_s \times \boldsymbol{V}_s^{(n-1)} + \mathcal{T}^{(n)}\boldsymbol{X}_s \tag{1.3}$$
$$\vdots$$

where the subscripts denote the partial differentiation with respect to the indicated variables and $\mathcal{T}^{(n)}$ is a function to be determined by the condition of the arclength parametrization: $\boldsymbol{V}_s^{(n)} \cdot \boldsymbol{X}_s = 0$. Equating $\boldsymbol{V}^{(1)}$ with $\boldsymbol{X}_t$, the first equation gives the LIE with appropriately rescaled time. Next, if we take $\boldsymbol{X}_t = \boldsymbol{V}^{(1)} + \epsilon\boldsymbol{V}^{(2)}$, ϵ some parameter, we recover the localized induction equation of a vortex filament with axial flow in the core [6, 7]. Recall that it is equivalent to the Hirota equation [8] which results from a summation of the first two terms of the NLS hierarchy as it should be (see also [9] and [10]). With this observation, it is tempting to pursue the summation procedure of vector fields of the LPH. Incidentally, families of evolution equations that preserve local geometric invariants are produced by combining finite terms of the LPH [11, 12].

The objective of the present investigation is to establish an evolution equation of a curve by summing up all of the infinite vector fields of the LPH and to disclose its properties.

In section 2, the summation procedure is implemented. We demonstrate that the resulting equation is equivalent to the Lund–Regge equation, which was derived as a model for the motion of a relativistic string in a constant external field [13]. In section 3, we rewrite our equation into the intrinsic form, that is, evolution equations for the curvature and the torsion. They are reduced to the sine–Gordon equation if the filament takes a specific value of torsion. In section 4, we seek exact solutions. By using the methods of Kida [14] and Fukumoto [15, 16], the whole family of filaments which travel steadily with no deformation are obtained. The behaviour of a soliton, a localized twist wave, is discussed in some detail. Each shape admits two propagating speeds of a twist, one of which has no counterpart in the LIE and is therefore peculiar to the infinite summation of the LPH. The similar is true for the whole class. Section 5 is devoted to a summary and conclusions.

2. Summation of Langer–Perline hierarchy

Consider the evolution equation of a curve obtained by summing up all of the terms of the LPH, namely

$$\boldsymbol{X}_t = \boldsymbol{V}^{(1)} + \epsilon\boldsymbol{V}^{(2)} + \epsilon^2\boldsymbol{V}^{(3)} + \cdots = \sum_{n=1}^{\infty}\epsilon^{n-1}\boldsymbol{V}^{(n)}. \tag{2.1}$$

Here the coefficient of each term is taken to be an integral power of some constant ϵ. This infinite summation is rather formal.

By virtue of the recursion relation (1.3), the resulting equation is expressed in a compact form:

$$\boldsymbol{X}_t = \boldsymbol{X}_s \times \boldsymbol{X}_{ss} - \epsilon\boldsymbol{X}_s \times \boldsymbol{X}_{ts} + \mathcal{T}\boldsymbol{X}_s \tag{2.2}$$

where

$$\mathcal{T} = \tfrac{1}{2}\epsilon\boldsymbol{X}_t \cdot \boldsymbol{X}_t + C(t) \tag{2.3}$$

with $C(t)$ being an arbitrary real function of t, and the condition $\boldsymbol{X}_s \cdot \boldsymbol{X}_s = 1$ is to be kept in view. The derivation of (2.3) is straightforward; we first differentiate both sides of (2.2) with respect to s, and thereafter take the inner product with $\boldsymbol{X}_s$. Using (2.2) again, we have $\mathcal{T}_s = \epsilon\boldsymbol{X}_{st} \cdot \boldsymbol{X}_t$, from which (2.3) follows. The second term on the right-hand side of (2.2) appears to be a perturbation to the LIE. However, it predominates in the time evolution in the sense that the first term is absorbed into the second one simply by the change of a variable $s \to s - t/\epsilon$. It deserves a mention that this structure is accommodated in the equation derived by Moore and Saffman [17] for the motion of a vortex filament with the effect of an axial flow in the core being taken into account. Notably, (2.2) and (2.3) assure not only arclength preservation but also conservation of the writhing number for a closed filament [18].

It is illuminating to rewrite (2.2) into an alternative form. By taking the exterior product with $\boldsymbol{X}_s$, (2.2) is converted into

$$\boldsymbol{X}_s \times \boldsymbol{X}_t = -\boldsymbol{X}_{ss} + \epsilon \boldsymbol{X}_{st}. \tag{2.4}$$

Introducing the new variables

$$\zeta = s \qquad \eta = 2t/\epsilon + s \tag{2.5}$$

we arrive at

$$\boldsymbol{X}_{\zeta\zeta} - \boldsymbol{X}_{\eta\eta} = -\frac{2}{\epsilon} \boldsymbol{X}_\zeta \times \boldsymbol{X}_\eta. \tag{2.6}$$

This equation, supplemented by two auxiliary conditions

$$\boldsymbol{X}_\zeta^2 + \boldsymbol{X}_\eta^2 = 1 - \epsilon C(t) \tag{2.7}$$

$$\boldsymbol{X}_\zeta \cdot \boldsymbol{X}_\eta = \tfrac{1}{2}\epsilon C(t) \tag{2.8}$$

is none other than the Lund–Regge equation [13]. It was given birth to as a byproduct of a unified theory of the Nambu string, a relativistic string, and the classical vortex filament. Notice that (2.7) and (2.8) differ from the original ones. To gain our expressions, it suffices to choose

$$x^0 = \frac{1}{2}(\zeta + \eta) + \frac{1}{\epsilon} \int^{\epsilon(\zeta-\eta)/2} \sqrt{1 - 2\epsilon C(t)}\, \mathrm{d}t$$

instead of $x^0 = \eta$ in equation (3.1) of [13]. Our equation meets the conditions (2.7) and (2.8), which is proved with no difficulty in the following way:

$$\boldsymbol{X}_\zeta^2 + \boldsymbol{X}_\eta^2 = \boldsymbol{X}_s^2 - \epsilon \boldsymbol{X}_s \cdot \boldsymbol{X}_t + \tfrac{1}{2}\epsilon^2 \boldsymbol{X}_t^2 = 1 - \epsilon C(t) \tag{2.9}$$

$$\boldsymbol{X}_\zeta \cdot \boldsymbol{X}_\eta = \tfrac{1}{2}\epsilon \boldsymbol{X}_t \cdot (\boldsymbol{X}_s - \tfrac{1}{2}\epsilon \boldsymbol{X}_t) = \tfrac{1}{2}\epsilon C(t). \tag{2.10}$$

Langer and Perline picked out a restricted class of curves, the BAL. Under this restriction, $C(t)$ disappears and the vector fields $\boldsymbol{V}^{(n)}$ are commutative with each other. It is worth noting that (2.6)–(2.8) are, in the case of $C = 0$, equivalent to the Lund–Regge–Pohlmeyer–Getmanov equations, a complexified sine–Gordon equation, which is solvable by the inverse scattering method [13, 19]. Probably, the condition of $C = 0$ is necessary in order for our equation to be completely integrable. Moreover, only in this case, (2.2) is reducible to the LIE, in its simplest form, in the limit of $\epsilon \to 0$. Hereafter, we assume that $C = 0$.

3. Intrinsic equations

We deduce the intrinsic form of (2.2) or (2.4) along the line of Hasimoto's procedure [3, 9]. Let us introduce the complex curvature ψ and a complex vector $\boldsymbol{N}$ defined by

$$\psi = \kappa \exp\left[\mathrm{i} \int^s \tau \, \mathrm{d}s\right] \qquad \boldsymbol{N} = (n + \mathrm{i}\boldsymbol{b}) \exp\left[\mathrm{i} \int^s \tau \, \mathrm{d}s\right] \tag{3.1}$$

where κ and τ are the curvature and the torsion, and $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ are the Frenet–Serret frame of a curve. The Frenet–Serret formulae are then written as

$$\boldsymbol{t}_s = -\tfrac{1}{2}(\psi^* \boldsymbol{N} + \psi \boldsymbol{N}^*) \qquad \boldsymbol{N}_s = -\psi \boldsymbol{t}. \tag{3.2}$$

Here the asterisk indicates complex conjugate. Using the identities $\boldsymbol{N} \cdot \boldsymbol{N}^* = 2$, $\boldsymbol{N} \cdot \boldsymbol{N} = \boldsymbol{N} \cdot \boldsymbol{t} = \boldsymbol{N}^* \cdot \boldsymbol{t} = 0$, the time derivatives of $\boldsymbol{t}$ and $\boldsymbol{N}$ can be generally expressed, by making use of some real function R and some complex function γ, as

$$\boldsymbol{t}_t = -\tfrac{1}{2}(\gamma^* \boldsymbol{N} + \gamma \boldsymbol{N}^*) \qquad \boldsymbol{N}_t = \mathrm{i}R\boldsymbol{N} + \gamma \boldsymbol{t}. \tag{3.3}$$

Differentiating (2.2) with respect to s, we get, after some algebra,

$$\gamma = -\mathrm{i}\psi_s + \mathrm{i}\epsilon\psi_t - (\tfrac{1}{2}\epsilon \boldsymbol{X}_t \cdot \boldsymbol{X}_t - \epsilon R)\psi. \tag{3.4}$$

The integrability condition $\boldsymbol{N}_{st} = \boldsymbol{N}_{ts}$ (or $\boldsymbol{t}_{st} = \boldsymbol{t}_{ts}$) requires

$$\psi_t = -\gamma_s + \mathrm{i}R\psi \tag{3.5}$$

$$R_s = \tfrac{1}{2}\mathrm{i}(\gamma\psi^* - \gamma^*\psi). \tag{3.6}$$

Plugging (3.4) into (3.6), we have

$$R_s = \tfrac{1}{2}|\psi|_s^2 - \tfrac{1}{2}\epsilon|\psi|_t^2. \tag{3.7}$$

On the other hand, using the identity $\gamma = -\boldsymbol{t}_t \cdot \boldsymbol{N}$, (3.6) leads to

$$R_s = \boldsymbol{t}_t \cdot \kappa \boldsymbol{b} = \boldsymbol{X}_{st} \cdot \boldsymbol{X}_t \tag{3.8}$$

the last equality coming from (2.2) and its spatial derivative. Equation (3.8) helps to simplify (3.4). It turns out that we may ignore the integration constant in R, being an arbitrary real function of t, because it can be absorbed into the phase factor of ψ without affecting the curve dynamics. Substitution of (3.4) and (3.7) into (3.5) yields, with the help of (3.8),

$$\psi_t = \mathrm{i}\psi_{ss} + \tfrac{1}{2}\mathrm{i}|\psi|^2\psi - \mathrm{i}\epsilon\left(\psi_{st} + \frac{1}{2}\psi\int^s |\psi|_t^2\,\mathrm{d}s\right). \tag{3.9}$$

In keeping with the procedure of infinite summation (2.1), the same equation is reached via use of the recursion operator associated with the NLS hierarchy [4, 5]. According to the form of this operator for the BAL, the indefinite integral in (3.9) is replaced by a definite integral:

$$\frac{1}{2}\left(\int_{-\infty}^{s} |\psi|_t^2\,\mathrm{d}s - \int_{s}^{\infty} |\psi|_t^2\,\mathrm{d}s\right). \tag{3.10}$$

Splitting (3.9) into the real and imaginary parts, we obtain

$$\kappa_t = -(2\kappa_s\tau + \kappa\tau_s) + \epsilon\left(\kappa_t\tau + \kappa\tau_t + \kappa_s\int^s \tau_t\,\mathrm{d}s\right) \tag{3.11}$$

$$\int^s \tau_t\,\mathrm{d}s = \frac{\kappa_{ss}}{\kappa} - \tau^2 + \frac{\kappa^2}{2} - \epsilon\left(\frac{\kappa_{st}}{\kappa} - \tau\int^s \tau_t\,\mathrm{d}s + \int^s \kappa\kappa_t\,\mathrm{d}s\right) \tag{3.12}$$

the later of which becomes, upon differentiating with respect to s,

$$\tau_t = \left(\frac{\kappa_{ss}}{\kappa}\right)_s - 2\tau\tau_s + \kappa\kappa_s - \epsilon\left[\left(\frac{\kappa_{st}}{\kappa}\right)_s - \tau_s\int^s \tau_t\,\mathrm{d}s - \tau\tau_t + \kappa\kappa_t\right]. \tag{3.13}$$

In a special case, (3.11) and (3.12) are collapsed into the sine–Gordon equation. In terms of the variables $\hat{t} = t$ and $\hat{s} = s + t/\epsilon$, they read

$$\kappa_{\hat{t}} + \frac{1}{\epsilon}\kappa_{\hat{s}} = \epsilon\left(\kappa_{\hat{t}}\tau + \kappa\tau_{\hat{t}} + \kappa_{\hat{s}}\int^{\hat{s}} \tau_{\hat{t}}\,\mathrm{d}\hat{s}\right) \tag{3.14}$$

$$\int^{\hat{s}} \tau_{\hat{t}}\,\mathrm{d}\hat{s} + \frac{1}{\epsilon}\tau = -\epsilon\left(\frac{\kappa_{\hat{s}\hat{t}}}{\kappa} - \tau\int^{\hat{s}} \tau_{\hat{t}}\,\mathrm{d}\hat{s} + \int^{\hat{s}} \kappa\kappa_{\hat{t}}\,\mathrm{d}\hat{s}\right). \tag{3.15}$$

The integral of torsion in the definition of (3.1) is an indefinite integral, and therefore a constant is at our disposal. If we set $\tau = 1/\epsilon$, the first equation is identically satisfied with a choice of the integration constant in such a way that $\int^{\hat{s}} \tau_{\hat{t}}\, \mathrm{d}\hat{s} = 1/\epsilon^2$.

For definiteness, we restrict our attention to the BAL. Their curvature vanishes at infinity. In view of (3.10), (3.15) becomes

$$\frac{\kappa_{\hat{s}\hat{t}}}{\kappa} + \frac{1}{2}\left(\int_{-\infty}^{\hat{s}} \kappa\kappa_{\hat{t}}\, \mathrm{d}\hat{s} - \int_{\hat{s}}^{\infty} \kappa\kappa_{\hat{t}}\, \mathrm{d}\hat{s}\right) = -\frac{1}{\epsilon^3}. \tag{3.16}$$

Following Nakayama *et al* [20], we define

$$\theta = \int_{-\infty}^{\hat{s}} \kappa\, \mathrm{d}\hat{s} \tag{3.17}$$

and prescribe the temporal evolution of κ as

$$\kappa_{\hat{t}} = -\frac{1}{\epsilon^3}\sin\theta. \tag{3.18}$$

Substituting from (3.17) and (3.18) and noting from (3.18) that $\sin\theta \to 0$ as $\hat{s} \to \pm\infty$, we find that (3.16) holds true. The consistency of (3.17) with (3.18) gives rise to

$$\theta_{\hat{s}\hat{t}} = -\frac{1}{\epsilon^3}\sin\theta. \tag{3.19}$$

4. Filament moving without change of form

The LIE admits solutions which express invariant shapes of a vortex filament moving steadily through a still fluid. This problem was successfully solved by Kida [14]. We make an attempt to search for this sort of exact solutions of our model.

We employ Kida's ansatz. Extending Hasimoto's idea [21], he considered that such a motion comprises three ingredients, namely, a translation with velocity V in a certain direction, say the z-direction, a rotation about the same axis with angular velocity Ω, and a slipping motion along itself with speed c_0. The resulting equation takes on the form

$$\boldsymbol{X}_t = -c_0\boldsymbol{X}_s + \Omega e_z \times \boldsymbol{X} + V e_z \tag{4.1}$$

where $\boldsymbol{e}_z$ is the unit vector in the z-direction, and c_0, Ω, V are all constants. The general solution of (4.1) is written, in Cartesian coordinates (X, Y, Z), as

$$X + \mathrm{i}Y = r(\xi)\exp[\mathrm{i}(\phi(\xi) + \Omega t)] \tag{4.2}$$

$$Z = z(\xi) + Vt \tag{4.3}$$

where

$$\xi = s - c_0 t \tag{4.4}$$

and $r(\xi)$, $\phi(\xi)$ and $z(\xi)$ are arbitrary functions of ξ. The determination of these functions requires a knowledge of the dynamical equation for the filament. Substitution of (4.1) into (2.2) yields

$$-c_0\boldsymbol{X}_\xi + \Omega\boldsymbol{e}_z \times \boldsymbol{X} + V\boldsymbol{e}_z = (1+\epsilon c_0)\boldsymbol{X}_\xi \times \boldsymbol{X}_{\xi\xi} - \epsilon\Omega[\boldsymbol{e}_z - (\boldsymbol{X}\cdot\boldsymbol{e}_z)\boldsymbol{X}_\xi] + \mathcal{T}\boldsymbol{X}_\xi \tag{4.5}$$

because of $\partial/\partial s = \partial/\partial\xi$. Multiplying (4.5) vectorially by $\boldsymbol{X}_\xi$, we are left with

$$(1+\epsilon c_0)\boldsymbol{X}_{\xi\xi} = -\boldsymbol{X}_\xi \times [\Omega\boldsymbol{e}_z \times \boldsymbol{X} + (V+\epsilon\Omega)\boldsymbol{e}_z]. \tag{4.6}$$

If we look upon ξ as the time and upon $(1+\epsilon c_0)$ as m/q, (4.6) is identifiable as the equation governing the motion of a charged particle with mass m and charge q in the magnetic field $-\Omega\boldsymbol{e}_z \times \boldsymbol{X} - (V+\epsilon\Omega)\boldsymbol{e}_z$ (cf Fukumoto [15, 16]).

This observation allows us to appeal to the technique of classical mechanics. We concentrate on the case of $1+\epsilon c_0 \neq 0$. The degenerate case of $1+\epsilon c_0 = 0$ will be treated in the appendix. The Lagrangian $\mathcal{L}$ for (4.6) is

$$\mathcal{L} = \frac{1+\epsilon c_0}{2}(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2) - \frac{V+\epsilon\Omega}{2} r^2\dot{\phi} + \frac{\Omega}{2} r^2 \dot{z} \tag{4.7}$$

where a dot denotes the differentiation in ξ, and the definitions (4.2) and (4.3) have been used. Inspection says that z and ϕ are both cyclic, and the first integrals are available at once:

$$\dot{z} + \frac{\hat{\Omega}}{2} r^2 = P \tag{4.8}$$

$$r^2\dot{\phi} - \frac{\hat{V}+\epsilon\hat{\Omega}}{2} r^2 = M \tag{4.9}$$

where P and M are integration constants, and

$$\hat{\Omega} = \frac{\Omega}{1+\epsilon c_0} \qquad \hat{V} = \frac{V}{1+\epsilon c_0}. \tag{4.10}$$

The conservation of kinetic energy of the particle is embodied by the arclength parametrization of the curve:

$$|\dot{\boldsymbol{X}}|^2 = \dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2 = 1. \tag{4.11}$$

The constants P and M are not independent. Taking the inner product of (4.5) with $\boldsymbol{X}_\xi$ and recalling the expression (2.3) of $\mathcal{T}$ with $C = 0$, we have

$$(1+\epsilon c_0)(\Omega r^2\dot{\phi} + V\dot{z}) = c_0 + \tfrac{1}{2}\epsilon(c_0^2 + V^2) + \tfrac{1}{2}\epsilon\Omega^2 r^2. \tag{4.12}$$

Substitution from (4.8) and (4.9) provides us with the constraint

$$\hat{\Omega} M + \hat{V} P = \hat{c_0} + \tfrac{1}{2}\epsilon(\hat{V}^2 - \hat{c_0}^2) \tag{4.13}$$

where

$$\hat{c_0} = \frac{c_0}{1+\epsilon c_0}. \tag{4.14}$$

The set of equations to be solved are (4.8), (4.9) and (4.11). They have the same form, except for the coefficients, as in the case of $\epsilon = 0$, so Kida's procedure is straightforwardly applicable. In the following, we present a brief description of the outline of integration.

Inserting (4.8) and (4.9), ϕ and z are eliminated from (4.11) to give

$$\dot{\rho}^2 = f(\rho) \tag{4.15}$$

where

$$\rho = r^2 \tag{4.16}$$

and

$$f(\rho) = -\hat{\Omega}^2\rho^3 + [4\hat{\Omega}P - (\hat{V}+\epsilon\hat{\Omega})^2]\rho^2 + 4[1 - P^2 - M(\hat{V}+\epsilon\hat{\Omega})]\rho - 4M^2. \tag{4.17}$$

Noting that $f(0) \leqslant 0$, we see that the realizable solution is available only when $f(\rho)$ has three real roots ρ_1, ρ_2, ρ_3 such that $\rho_3 \leqslant 0 \leqslant \rho_2 \leqslant \rho_1$. The solution ranges over $\rho_2 \leqslant \rho \leqslant \rho_1$.† The solution of (4.15) is then expressible, in terms of the Jacobian elliptic function, as

$$r^2 = \rho_1 - (\rho_1 - \rho_2)\mathrm{sn}^2(\hat{\xi}|k) \tag{4.18}$$

† The case of $\hat{\Omega} = 0$ is separately treated in the same manner as in [14] and the solution is a uniform helix.

where

$$\hat{\xi} = \tfrac{1}{2}(\rho_1 - \rho_3)^{1/2}|\hat{\Omega}|\xi \tag{4.19}$$

and the elliptic modulus is

$$k = \left(\frac{\rho_1 - \rho_2}{\rho_1 - \rho_3}\right)^{1/2}. \tag{4.20}$$

Upon substitution of (4.18), (4.8) and (4.9) are immediately integrated to give

$$z = \left(P - \frac{1}{2}\hat{\Omega}\rho_3\right)\xi - \frac{\hat{\Omega}}{|\hat{\Omega}|}(\rho_1 - \rho_3)^{1/2}E(\hat{\xi} \mid k) + z_0 \tag{4.21}$$

$$\phi = \frac{1}{2}(\hat{V} + \epsilon\hat{\Omega})\xi + \frac{2M}{\rho_1(\rho_1 - \rho_3)^{1/2}|\hat{\Omega}|}\Pi\left(\hat{\xi} \mid \frac{\rho_1 - \rho_2}{\rho_1}, k\right) + \phi_0 \tag{4.22}$$

where z_0 and ϕ_0 are integration constants, and

$$E(u \mid k) = \int_0^u \mathrm{d}n^2(u' \mid k)\,\mathrm{d}u' \tag{4.23}$$

and

$$\Pi(u \mid l, k) = \int_0^u \frac{\mathrm{d}u'}{1 - l\,\mathrm{sn}^2(u' \mid k)} \tag{4.24}$$

are the incomplete elliptic integrals of second and third kinds, respectively.

As is evident from (4.8), (4.9) and (4.11), the filament form depends upon the four parameters $\hat{\Omega}$, $\hat{V} + \epsilon\hat{\Omega}$, P and M. Among them, three are relevant, since the roots of the cubic equation $f(\rho)$ determine the form. Kida's catalogue cover the whole family, so we skip the detail of classification of the shapes.

However, it is remarkable that even if these parameters are specified and thus the form is fixed at some instant, say at $t = 0$, (4.13) admits two solutions for $\hat{c}_0$,

$$\hat{c}_0 = \frac{1}{\epsilon} \pm \sqrt{\frac{1}{\epsilon^2} - \frac{2}{\epsilon}\left(\hat{\Omega}M + \hat{V}P\right) + \hat{V}^2}. \tag{4.25}$$

One of them (the minus sign) recovers that of the LIE in the limit of $\epsilon \to 0$. The other branch, which increases unboundedly as ϵ is decreased, is acquired as the result of the infinite summation of the LPH. To put it in another way, it is missing if the summation is truncated at a finite order.

This situation is lucidly exemplified by the one-soliton solution. It occurs in the limit of $k \to 1$. In view of (4.20), $\rho_2 = \rho_3 = 0$ in this limit and (4.17) produces

$$M = 0 \tag{4.26}$$

$$1 - P^2 - M(\hat{V} + \epsilon\hat{\Omega}) = 0 \tag{4.27}$$

and the non-negative root of $f(\rho)$ is

$$\rho_1 = \frac{4P}{\hat{\Omega}} - \frac{(\hat{V} + \epsilon\hat{\Omega})^2}{\hat{\Omega}^2}. \tag{4.28}$$

The first two together give $P^2 = 1$. Inspection shows that we may put

$$P = 1 \qquad \hat{\Omega} > 0 \tag{4.29}$$

without loss of generality. Thanks to (4.26) and (4.29), (4.13) is factorized to yield

$$\hat{c}_0 = \hat{V} \qquad \frac{2}{\epsilon} - \hat{V}. \tag{4.30}$$

It is the second one that has no counterpart in the case of the LIE. We name the soliton solution with $\hat{c}_0 = \hat{V}$ the slow soliton and that with $\hat{c}_0 = 2/\epsilon - \hat{V}$ the fast soliton. This is because, for the latter, $\hat{c}_0$ and thus c_0 become infinity in the limit of $\epsilon \to 0$. It is reminiscent of the sound waves in the incompressible limit.

To facilitate the comparison with Hasimoto's expression [3], write

$$\hat{\Omega} = \tau_0^2 + \nu^2 \qquad \hat{V} + \epsilon\hat{\Omega} = 2\tau_0 \tag{4.31}$$

using constants τ_0 and ν. It follows from (4.28) that $\rho_1^{1/2} = 2\nu/(\tau_0^2 + \nu^2)$. Taking the limit of $k \to 1$ in (4.18)–(4.24) and substituting them into (4.2) and (4.3), we get the one-soliton solution in the form

$$X + \mathrm{i}Y = \frac{2\nu}{\tau_0^2 + \nu^2}\,\mathrm{sech}[\nu(s - c_0 t)]\,\mathrm{e}^{\mathrm{i}\Phi}. \tag{4.32}$$

Here, for the slow soliton,

$$c_0 = \frac{2\tau_0 - \epsilon(\tau_0^2 + \nu^2)}{(1 - \epsilon\tau_0)^2 + \epsilon^2\nu^2} \tag{4.33}$$

$$\Phi = \tau_0 s + \frac{\nu^2 - \tau_0^2 + \epsilon\tau_0(\tau_0^2 + \nu^2)}{(1 - \epsilon\tau_0)^2 + \epsilon^2\nu^2} t + \phi_0 \tag{4.34}$$

$$Z = s - \frac{2\nu}{\tau_0^2 + \nu^2}\tanh[\nu(s - c_0 t)] + z_0 \tag{4.35}$$

and for the fast one,

$$c_0 = \frac{2\tau_0 - \epsilon(\tau_0^2 + \nu^2) - 2/\epsilon}{(1 - \epsilon\tau_0)^2 + \epsilon^2\nu^2} \tag{4.36}$$

$$\Phi = \tau_0 s + \frac{1}{\epsilon}\frac{(1 - \epsilon\tau_0)[2\tau_0 - \epsilon(\tau_0^2 + \nu^2)]}{(1 - \epsilon\tau_0)^2 + \epsilon^2\nu^2} t + \phi_0' \tag{4.37}$$

$$Z = s - \frac{2\nu}{\tau_0^2 + \nu^2}\tanh[\nu(s - c_0 t)] + \frac{2}{\epsilon}t + z_0' \tag{4.38}$$

with ϕ_0, z_0, ϕ_0' and z_0' being arbitrary constants. It is informative to add that

$$\Omega = \pm\frac{\tau_0^2 + \nu^2}{(1 - \epsilon\tau_0)^2 + \epsilon^2\nu^2} \qquad V = \pm\frac{2\tau_0 - \epsilon(\tau_0^2 + \nu^2)}{(1 - \epsilon\tau_0)^2 + \epsilon^2\nu^2} \tag{4.39}$$

with the plus sign being chosen for the slow soliton and the minus sign for the fast one. The soliton solution is a curve of constant torsion and the torsion in the above expression agrees with τ_0. The curvature is

$$\kappa = 2\nu\,\mathrm{sech}[\nu(s - c_0 t)]. \tag{4.40}$$

They fulfil (3.11) and (3.12). Note that a kink solution of (3.19) coincides with the slow soliton (4.40) with $\tau_0 = 1/\epsilon$ and c_0 specified by (4.33).

The slow soliton indeed reduces to the Hasimoto soliton if the perturbation is switched off ($\epsilon = 0$). The slipping speed c_0 of (4.33) and the coefficient of t in (4.34) are obtainable from those of the Hasimoto soliton simply by the replacement $\sigma^2 \mapsto \sigma^2/(1 - \epsilon\sigma)$ with $\sigma = \tau_0 + \mathrm{i}\nu$. Furthermore, (4.32)–(4.35) are reduced, up to $O(\epsilon)$, to a soliton on a vortex filament with axial velocity [6, 7, 22]. By contrast, the fast soliton is not accessible by perturbing the Hasimoto soliton.

The origin of the existence of the fast soliton is traced to the symmetry associated with the Lund–Regge equation. The salient feature of (2.6)–(2.8) is that they are invariant if the parameters ζ and η are interchanged. If (ζ, η) defined by (2.5) are used in place of (s, t), both the slow and fast solitons are cast into the solutions of the Lund–Regge equation, being equivalent to the one-soliton solution obtained by Sym *et al* [23]. We can confirm that the interchange of ζ and η converts the slow soliton into the fast one.

This situation carries over to the general solution (4.18)–(4.24) with the double-valued $\hat{c}_0$ given by (4.25). The minus sign corresponds to the slow mode and the plus sign to the fast mode. Owing to the form of $\hat{c}_0$, the differences of Ω and V between the two modes arise only in their signs, in the same way as (4.39). After some manipulations, we realize that these facts are closely tied with the interchangeability of ζ and η.

5. Conclusions

In this paper, we have highlightened some aspects of the Langer–Perline hierarchy that show up when the summation is extended to the infinite order. The recursion operator of the LPH renders it feasible.

We have verified that the resulting equation is reducible to the Lund–Regge equation. The application of Hasimoto's scheme brings in an intrinsic equation for the complex curvature in the form of an integrodifferential evolution equation. It has the same form as the equation obtained when the nonlinear Schrödinger hierarchy is summed up to the infinite order. Splitting it into the real and imaginary parts, we reach a generalization of the Betchov–Da Rios equations. In a special case, they are simplified to the sine–Gordon equation, being in accord with Lund and Regge's observation [13].

Our model possesses exact solutions of the same type as Kida derived, namely, the invariant forms of a filament steadily rotating and translating in the three-dimensional space. The shape remains unaltered from Kida's solution, but a profound difference makes its appearance in the movement. Given the shape, the travelling and rotating speeds are not uniquely determined. Instead, there are two kinds, one of which has a link with the solution of the LIE. The other is novel, because the speeds diverge in the limit that the model equation tends to the LIE. The symmetry of the Lund–Regge equation with respect to the interchange of the parameters underpins the existence of the new mode.

When we make a mathematical model to mimic natural phenomena, a common tactic is to invoke a perturbation-expansions technique. Usually, on account of difficulty, we cannot help truncating the expansions at a finite order in powers of a small parameter. However, it is probable that there are modes that cannot be captured without completing the expansions to the infinite order. The analysis of section 4 reveals that our model provides us with an example to illustrate the insufficiency of finite truncation. This inspires us to look into the relevance of our model with some natural phenomena.

We can show that (2.2) or (2.4) serves as a model for the motion of a vortex filament having irrotational jet in the core with the density larger than that of the surrounding fluid. The vorticity is concentrated in the cylindrical sheet surrounding the core and otherwise the flow is irrotational. The effect of the gravity force is ignored. The velocity of the centre line of the core is obtained in powers of a small parameter, the ratio of the core to the curvature radii. Under the assumption of local induction, it takes the same form as (2.2). The second term comes from the internal jet and, when the density of the internal fluid is larger, it balances with the leading term. The description of the detail is postponed to a subsequent paper.

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Appendix. Degenerate case

When $1 + \epsilon c_0 = 0$, (4.6) becomes

$$\Omega \boldsymbol{e}_z \times \boldsymbol{X} + (V + \epsilon\Omega)\boldsymbol{e}_z = A\boldsymbol{X}_\xi \tag{A.1}$$

where A is a constant. It is determined by taking the inner product of (4.5) with $\boldsymbol{X}_\xi$, resulting in

$$A = \pm\frac{1}{\epsilon}\sqrt{1 + \epsilon^3\Omega(2V + \epsilon\Omega)}. \tag{A.2}$$

Substitution of the solution of (A.1) into (4.2) and (4.3) gives rise to a uniform helix:

$$X + \mathrm{i}Y = r_0 \exp\left\{\mathrm{i}\left[\frac{\Omega s}{A} + \Omega\left(1 + \frac{1}{\epsilon A}\right)t + \phi_0\right]\right\} \tag{A.3}$$

$$Z = \frac{V + \epsilon\Omega}{A}s + \left[V\left(1 + \frac{1}{\epsilon A}\right) + \frac{\Omega}{A}\right]t + z_0 \tag{A.4}$$

where r_0, ϕ_0 and z_0 are arbitrary constants.

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